AS-2780 B.Sc. (Physics) V^{th} Sem

Mathematical Physics.

Section A

Ans. ① (i) (a) 0

(ii) (a) $\frac{a}{s^2+a^2}$, $(s > ia)$,

(iii) (a) Real

(iv) (d) All of these

(v) (d) 4

(vi) (a) ordinary

(vii) (c) $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

(viii) (b) $y = \rho \sin\phi$

(ix) (b) $\hat{i}$

(x) (c) $1, \rho, 1$

Section B.

(2)

Ans. (2) The spherical coordinate system consists of three coordinates r, θ and ϕ. The coordinates of a point P in spherical coordinate system is (r, θ, ϕ).

It consists of.

(i) concentric spheres about origin O

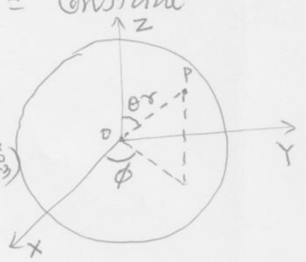
$$r = \sqrt{x^2+y^2+z^2} = \text{Constant}$$

(ii) Right circular cones about z-axis with vertex at the origin O

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \text{Constant}$$

(iii) Half planes through the z-axis

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{Constant}$$



In curvilinear coordinates, the divergence of a vector $\vec{A}$ $(\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(h_1A_2h_3) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

and curl of a vector $\vec{A}$

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$\therefore$ In spherical polar coordinates (r, θ, ϕ)

$$\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\phi\hat{e}_\phi. \qquad u_1 = r,\ u_2 = \theta,\ u_3 = \phi$$

$h_1 = h_r = 1$

$h_2 = h_\theta = r$

$h_3 = h_\phi = r\sin\theta.$

$$\therefore \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial(A_\phi)}{\partial\phi}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\left[\left\{\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)\right\}\hat{e}_r + \left\{\frac{\partial}{\partial\phi}(A_r) - \frac{\partial}{\partial r}(r\sin\theta A_\phi)\right\}r\hat{e}_\theta + \left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right\}r\sin\theta\,\hat{e}_\phi\right]$$

Ans(3) (i) $\frac{dy}{dx} = 3(y+2x)+1, \quad y(0)=0$ ③

$$\frac{dy}{dx} = 3y+6x+1$$

$$\frac{dy}{dx} - 3y = 6x+1$$

Linear diff. eqn.

$$I.F = e^{\int -3\,dx}$$

Sol. is $y e^{\int -3\,dx} = \int (6x+1)\cdot e^{\int -3\,dx}\cdot dx + C$

$$y e^{-3x} = \int (6x+1) e^{-3x} dx + C$$

$$= \int 6x e^{-3x} dx + \int e^{-3x} dx + C$$

$$= 6x \frac{e^{-3x}}{-3} - \int 6 \cdot \frac{e^{-3x}}{-3} dx + \int e^{-3x} dx + C$$

$$= -2x e^{-3x} + 2\int e^{-3x} dx + \int e^{-3x} dx + C$$

$$= -2x e^{-3x} + 3 \cdot \frac{e^{-3x}}{-3} + C$$

$$\therefore \quad y + 2x + 1 = C e^{3x}$$

When $x=0, \quad y(0)=0 \implies C=1$

$$\therefore \quad y+2x+1 = e^{3x}$$

(ii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}$

$$(D^2+3D+2)y = e^{2x}$$

Auxiliary eqn is $m^2+3m+2=0$

$$\implies m = -2, -1$$

$$\therefore \quad C.F = C_1 e^{-x} + C_2 e^{-2x}$$

$$P.I = \frac{1}{D^2+3D+2} e^{2x} = \frac{1}{2^2+3\cdot 2+2} e^{2x} = \frac{1}{12} e^{2x}$$

Solution is $y = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{12} e^{2x}$

Ans (4) Laguerre differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0 \quad --- (1)$$

$x=0$ is a non-essential singularity or removable singularity. So $x=0$ is a regular point. So solution of (1) can be written as a series expanded around $x=0$.

Let the series sol. be

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{K+\lambda} \quad --- (2) \qquad a_0 \neq 0$$

$$\therefore \quad \frac{dy}{dx} = \sum_{\lambda=0}^{\infty} a_\lambda (K+\lambda) x^{K+\lambda-1} \quad --- (3)$$

and

$$\frac{d^2y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (K+\lambda)(K+\lambda-1) x^{K+\lambda-2} \quad --- (4)$$

Substituting these in (1) we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (K+\lambda)^2 x^{K+\lambda-1} - \sum_{\lambda=0}^{\infty} a_\lambda (K+\lambda-n) x^{K+\lambda} = 0 \quad --- (5)$$

Equating the coefficient of lowest power of x i.e x^{K-1} equal to zero, we get indicial equation

$$a_0 K^2 = 0 \quad --- (6)$$

$\because$ $a_0 \neq 0$ so we must have $K=0$.

Equating to zero the coefficient of x^{K+j}, we get

$$a_{j+1}(K+j+1)^2 - a_j (K+j-n) = 0$$

$$\therefore \quad a_{j+1} = \frac{K+j-n}{(K+j+1)^2} a_j \quad --- (7)$$

For $K=0$,

$$a_{j+1} = \frac{j-n}{(j+1)^2} a_j$$

$$\therefore \quad a_1 = -\frac{n}{1^2} a_0 = (-1)\frac{n}{1^2}$$

$$a_2 = -\frac{(n-1)}{2^2} a_1 = (-1)^2 \cdot \frac{n(n-1)}{1^2 \cdot 2^2} a_0 = (-1)^2 \frac{n(n-1)}{(\lfloor 2)^2} a_0$$

$$a_3 = -\frac{(n-2)}{3^2} a_2 = (-1)^3 \frac{n(n-1)(n-2)}{(\lfloor 3)^2} a_0$$

$$\vdots$$

$$a_r = (-1)^r \frac{n(n-1)(n-2) \cdots (n-r+1)}{(\lfloor r)^2} a_0$$

$\therefore$ solution for $K = 0$ is

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r + \cdots$$

$$= a_0 \Big[1 + (-1) \frac{nx}{(\lfloor 1)^2} + (-1)^2 \frac{n(n-1)}{(\lfloor 2)^2} x^2 + \cdots$$

$$\cdots + (-1)^r \frac{n(n-1) \cdots (n-r+1)}{(\lfloor r)^2} x^r + \cdots \Big]$$

$$= a_0 \sum_{r=0}^{\infty} (-1)^r \frac{n(n-1) \cdots (n-r+1)}{(\lfloor r)^2} x^r$$

$$= a_0 \sum_{r=0}^{\infty} (-1)^r \frac{\lfloor n}{\lfloor n-r \, (\lfloor r)^2} x^r \quad \cdots \text{(9)}$$

If a n is a +ive integer and $a_0 = 1$, then the series terminates after n^{th} term. The solution then is said to be Laguerre polynomial of degree n.

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{\lfloor n}{(\lfloor r)^2 \lfloor n-r} x^r \quad \cdots \text{(10)}$$

The solution of Laguerre equation for +ive n integer is $y = A\, L_n(x)$.

Ans. (5) $A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$

The characteristic equation of the matrix is

$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)\left[(4-\lambda)(-3-\lambda)+8\right] + 7\left\{(-1)\cdot 2\right] + (-5)\left[-2(4-\lambda)\right] = 0$

or, $(5-\lambda)\left[-12-4\lambda+3\lambda+\lambda^2+8\right] - 14 + 40 - 10\lambda = 0$

or, $-60 - 20\lambda + 15\lambda + 5\lambda^2 + 40 + 12\lambda + 4\lambda^2 - 3\lambda^2 - \lambda^3$
$+ 8\lambda + 26 - 10\lambda = 0$

or, $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

or, $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

or, $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore$ eigen values are $\lambda = 1, 2, 3$

$\therefore$ the required diagonal matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Ans ⑥ $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$

The characteristic equation of matrix is

$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)\left[(4-\lambda)(5-\lambda)\right] - 2\left[0 + 2(4-\lambda)\right] = 0$

or, $(2-\lambda)(4-\lambda)(5-\lambda) - 4(4-\lambda) = 0$

or, $(4-\lambda)\left[(2-\lambda)(5-\lambda) - 4\right] = 0$

or, $(4-\lambda)\left[10 - 7\lambda + \lambda^2 - 4\right] = 0$

or, $(4-\lambda)\left[6 - 7\lambda + \lambda^2\right] = 0$

or, $(4-\lambda)(6-\lambda)(1-\lambda) = 0$

$\therefore$ eigen values are $\lambda = 1, 4, 6$

Now from $[A - \lambda I][X] = [0]$

we get $(2-\lambda)x_1 - 2x_3 = 0$

$(4-\lambda)x_2 = 0$

$-2x_1 + (5-\lambda)x_3 = 0$

$\therefore$ for $\lambda = 1$, $X = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ Normalized eigen vector $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 4$, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ Normalized eigen vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 6$, $X = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ Normalized eigenvector $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Ans ⑦ $f(x) = x^2$ in the interval $(-\pi, \pi)$ (8)

By Fourier theorem

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left\{x^2 \cdot \frac{\sin nx}{n}\right\}_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2x}{n} \cdot \sin nx\,dx\right]$$

$$= -\frac{2}{\pi n}\int_{-\pi}^{\pi} x \sin nx\,dx = -\frac{2}{\pi n}\left[\left\{-x\frac{\cos nx}{n}\right\}_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2}{\pi n^2}\left[\pi \cos n\pi + \pi \cos(-n\pi)\right] - \frac{2}{\pi n^2}\left[\frac{\sin n\pi}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{n^2}\left[(-1)^n + (-1)^n\right] = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$$= 0$$

$$\therefore\ x^2 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right]$$

Ans (8) First shifting property of the Laplace transform ⑨

If $f(s)$ is the Laplace transform of $F(t)$ then the Laplace transform of $e^{at} F(t)$ is $f(s-a)$.

$$\because \; L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

$$\therefore \; L\{e^{at} F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} F(t)\,dt$$

$$= f(s-a)$$

$$F(t) = e^{4t} \sinh(4t)$$

$$L\{\sinh(4t)\} = \frac{4}{s^2-4^2} \quad , \; s>4$$
$$= f(s)$$

$$\therefore \; L\{e^{4t} \sinh(4t)\} = f(s-4)$$

$$= \frac{4}{(s-4)^2 - 4^2}$$

$$= \frac{4}{s^2-8s+16-16}$$

$$= \frac{4}{s^2-8s}$$